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SOLVING NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS BY ADOMIAN DECOMPOSITION METHOD AND VARIATIONAL ITERATION METHOD

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Abstract

In this paper, a new kind of Numerical technique for a non-linear problem called the Adomian decomposition method and Variation iteration method are described and used to give exact solution for some well-known non-linear problem. In this method, the problems are initially approximated with possible unknowns. Then a correction functional is constructed by a general Lagrange multiplier, which can be identified optimally via the variational theory.

Key Words and Phrases : *Adomian decomposition method, Non-linear equation, Riccati equation, Successive approximation, Variational iteration method.*

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1. Introduction

The Adomian decomposition method(ADM) [1] is a semi-analytical method for solving ordinary and partial non-linear differential equations. The method was developed from the 1970s to 1990s by George Adomian. In 1978, Inokuti et al. [4] proposed a general Lagrange multiplier method to solve non-linear problems, which was first proposed to solve problems in quantum mechanics. The main feature of the method is described, the solution of a mathematical problem with linearization assumption is used as initial approximation or trial-function, then a more highly precise approximation [8] at some special point can be obtained.

2. Adomian Decomposition Method

For solving differential or integral equations [5, 6], solution are usually obtained as exact solutions defined in closed form expressions, or as series solutions normally obtained from concrete problems. To apply the Adomian decomposition method [1] for such non-linear ordinary differential equations[5, 9] we consider the general equation of the following form:

$$Ly + Ry + F(y) = g(x), \tag{1}$$

where the differential operator L may be considered as the highest order derivative in the equation, R is the remainder of the differential operator, $F(y)$ expresses the non-linear terms and $g(x)$ is an inhomogeneous term. If L is a first order operator given by by

$$L = \frac{d}{dx}, \tag{2}$$

then we assume that L is invertible and the inverse operator L^{-1} is given by

$$L^{-1}(\cdot) = \int_0^x (\cdot)dx \tag{3}$$

so that

$$L^{-1}Ly = y(x) - y(0). \tag{4}$$

However, if L is a second order differential operator given by

$$L = \frac{d^2}{dx^2}, \tag{5}$$

the inverse operator L^{-1} is regarded a two-fold integration operator defined by

$$L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)dxdx, \tag{6}$$

which means that

$$L^{-1}Ly \quad = \quad y(x) - y(0) - xy'(0). \tag{7}$$

In a parallel manner, if L is a third order differential operator, we can easily show that

$$L^{-1}Ly \quad = \quad y(x) - y(0) - xy'(0) - \frac{1}{2!}x^2y''(0) \tag{8}$$

For higher order operators, we can easily define the related inverse operators in a similar way.
Applying L^{-1} to both sides of (1) gives

$$y(x) \quad = \quad \Psi_0 - L^{-1}g(x) - L^{-1}Ry - L^{-1}F(y), \tag{9}$$

where

$$\Psi_0 = \begin{cases} y(0), & \text{for } L = \dfrac{d}{dx}, \\ y(0) + xy'(0), & \text{for } L = \dfrac{d^2}{dx^2}, \\ y(0) + xy'(0) + \frac{1}{2!}x^2y''(0), & \text{for } L = \dfrac{d^3}{dx^3}, \\ y(0) + xy'(0) + \frac{1}{2!}x^2y''(0) + \frac{1}{3!}x^3y'''(0), & \text{for } L = \dfrac{d^4}{dx^4}, \\ y(0) + xy'(0) + \frac{1}{2!}x^2y''(0) + \frac{1}{3!}x^3y'''(0) + \frac{1}{4!}x^4y^4(0), & \text{for } L = \dfrac{d^5}{dx^5}. \end{cases} \tag{10}$$

and so on. The Adomin decomposition method[1] admits the decomposition of y into an infinite series of components

$$y(x) \quad = \quad \sum_{n=0}^{\infty} y_n \tag{11}$$

and the non-linear term $F(y)$ be equated to an infinite series of polynomials

$$F(y) \quad = \quad \sum_{n=0}^{\infty} A_n \tag{12}$$

where A_n, are the Adomian polynomials. Substituting (12),and (11) into (9) gives

$$\sum_{n=0}^{\infty} y_n \quad = \quad \Psi_0 - L^{-1}g(x) - L^{-1}R\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{13}$$

The various components y_n of the solution y can be easily determined by the recursive relation

$$y_0 = \Psi_0 - L^{-1}(g(x)),$$

$$y_{k+1} = -L^{-1}(Ry_k) - L^{-1}(A_k), \quad k \geq 0 \tag{14}$$

Consequently, the first few components can be written as

$$\begin{aligned} y_0 &= \Psi_0 - L^{-1}g(x) \\ y_1 &= -L^{-1}(Ry_0) - L^{-1}(A_0), \\ y_2 &= -L^{-1}(Ry_1) - L^{-1}(A_1), \\ y_3 &= -L^{-1}(Ry_2) - L^{-1}(A_2), \\ y_4 &= -L^{-1}(Ry_3) - L^{-1}(A_3) \end{aligned}$$

Having determined the components y_n, $n \geq 0$, the solution y in a series form follows immediately. As stated before, the series may be summed to provide the solution in a closed form. However, for concrete problems, the n-term partial sum

$$\Phi_n = \sum_{k=0}^{n-1} y_k \tag{15}$$

may be used to give the approximate solution.

3. Example: Riccati differential equation

Consider first order non-linear differential equation

$$u' = 1 - t^2 + u^2, \quad u(0) = 0 \tag{16}$$

On applying the inverse operator L^{-1} we obtain

$$u = t - \frac{1}{3}\, u^3 + L^{-1}(u^2) \tag{17}$$

Using the decomposition series $u(t)$ and the polynomial representation for u^2 give

$$\sum_{n=0}^{\infty} u_n(t) = t - \frac{1}{3}\, u^3 + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{18}$$

It is important to point out the modified decomposition method is recommended here. In this approach we split the polynomial $t-\frac{1}{3}\,u^3$ into two part, namely, t will be assigned to the zero-th component u_0 , and $-\frac{1}{3}\,u^3$ that will be assigned to the component u_1 among other other terms. In this case, we use a modified recursive relation to accelerate the convergence[2] of the solution. The modified recursive relation is defined by

$$\left.\begin{aligned} u_0 &= t, \\ u_1 &= -\frac{1}{3}\,u^3 + L^{-1}\left(A_0\right), \\ u_{k+2} &= L^{-1}\left(A_{k+1}\right), \quad k \geq 0. \end{aligned}\right\} \tag{19}$$

Consequently, the first few components are given by

$$\left.\begin{aligned} u_0 &= t, \\ u_1 &= -\frac{1}{3}\,u^3 + L^{-1}\left(A_0\right) = -\frac{1}{3}\,u^3 + L^{-1}\left(u_0^2\right) = 0, \\ u_{k+2} &= 0, \quad k \geq 0. \end{aligned}\right\} \tag{20}$$

The exact solution is given by $u(t) = t$.

4. Variational iteration Method

The Variational iteration method (VIM)[3, 4, 9] gives rapidly convergent successive approximations of the exact solution if an exact solution exists. The obtained approximations by this method are of high accuracy level even if few iteration used. As introduced before,the method employs the correction functional

$$u_{n+1}(x,t) \;=\; u_n(x,t) + \int_0^t \lambda(\xi)\left(Lu_n(x,\xi) + N\tilde{u_n}(x,\xi) - g(x,\xi)\right)d\xi,\ n \geq 0 \tag{21}$$

for the differential equation

$$Lu + Ru \;=\; g(x,t). \tag{22}$$

The Lagrange's multiplier $\lambda(\xi)$ should be determined first. This value of λ allows us to determine the successive approximations $u_{n+1}(x,t),\ n \geq 0$ of the solution $u(x,t)$ by using any zero-th approximation $u_0(x,t)$. The exact solution may be obtained by using $u(x,t) = lim_{n\to\infty} u_n(x,t)$.

To find λ we sometimes use

$$\left.\begin{aligned}\int \lambda(\xi)u_n'(\xi)d\xi &= \lambda(\xi)u_n(\xi) - \int \lambda'(\xi)u_n(\xi)d\xi,\\ \int \lambda(\xi)u_n''(\xi)d\xi &= \lambda(\xi)u_n'(\xi) - \lambda'(\xi)u_n(\xi) + \int \lambda''(\xi)u_n(\xi)d\xi\end{aligned}\right\} \tag{23}$$

and so on. The method has been used so far for handling linear problems only.

The variational iteration method [3, 4, 9] will be used to handle non-linear problems in a manner similar to that used before for linear problems. The method facilitates the computational work for non-linear problems compared to Adomian method. Unlike Adomian decomposition method, the variational iteration method does not require specific treatment for non-linear operators. There is no need for Adomian polynomials that require a huge size of computational work. Moreover, the variational iteration method does not require specific assumption or restrictive conditions as required by other method such as perturbation techniques. The effectiveness and the efficiency of the method can be confirmed by discussing the following non-linear ordinary differential equations.

5. Example: Riccati differential equation

Consider first order non-linear differential equation

$$u' = 1 - t^2 + u^2, \quad u(0) = 0 \tag{24}$$

The correctional functional for equation (24) is

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\xi)\left(u_n'(\xi) - \tilde{u_n}^2(\xi) + \xi^2 - 1\right)d\xi \tag{25}$$

The stationary conditions

$$\left.\begin{aligned}1 + \lambda\,|_{\xi=t} &= 0\\ \lambda'\,|_{\xi=t} &= 0\end{aligned}\right\} \tag{26}$$

follow immediately, this in turn give $\lambda = -1$.

Substituting this value of the $\lambda = -1$ into the functional (25), gives the iteration formula

$$u_{n+1}(t) = u_n(t) - \int_0^t \left(u_n'(\xi) - u_n^2(\xi) + \xi^2 - 1\right)d\xi, \qquad n \geq 0. \tag{27}$$

We can select $u_0(t) = u(0) = 0$ from the given condition. Proceeding as before we obtain the successive approximations

$$\begin{aligned}
u_0(t) &= 0, \\
u_1(t) &= 0 - \int_0^t \left(u_0'(\xi) - u_0^2(\xi) + \xi^2 - 1\right) d\xi = t - \frac{1}{3}\, t^3, \\
u_2(t) &= t - \frac{1}{3}\, t^3 - \int_0^t \left(u_1'(\xi) - u_1^2(\xi) + \xi^2 - 1\right) d\xi \\
&= t - \frac{1}{3}\, t^3 + \frac{1}{3}\, t^3 - \frac{2}{15}\, t^5 + \frac{1}{63}\, t^7, \\
u_3(t) &= t - \frac{1}{3}\, t^3 + \frac{1}{3}\, t^3 - \frac{2}{15}\, t^5 + \frac{1}{63}\, t^7 - \int_0^t \left(u_2'(\xi) - u_2^2(\xi) + \xi^2 - 1\right) d\xi \\
&= t - \frac{1}{3}\, t^3 + \frac{1}{3}\, t^3 - \frac{2}{15}\, t^5 + \frac{1}{63}\, t^7 + \cdots, \\
&\vdots \\
u_n(t) &= t - \frac{1}{3}\, t^3 + \frac{1}{3}\, t^3 - \frac{2}{15}\, t^5 + \frac{1}{63}\, t^7 - \frac{1}{63}\, t^7 + \cdots.
\end{aligned}$$

It is clear that the more terms vanish in the limit, and the exact solution is $u(t) = t$.

6. Conclusions

In this paper, we have worked-out exact solution for Riccati differential equation by using Adomin decomposition method and variational iteration method with some numerical techniques. The results show the following:

1. A correctional functional can be easily constructed by a general Lagrange multiplier, and the multiplier can be optimally identified by variational theory. The application of restricted variations in correction functional makes it much easier to determine the multiplier.
2. The initial approximation can be freely selected with unknown constants, which can be determined via various methods.
3. The preexaminations obtained by this method are not only for small parameter, but also for very large parameter, furthermore their first-order approximations are of extreme accuracy.
4. Comparison with Adomian decomposition method reveals that the approximations obtained by the proposed method converge to its exact solution faster than those of Adomian's.

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